Intro Video: Section 4.5
Curve Sketching

Math F251X: Calculus 1

We have computers. Why even bother?

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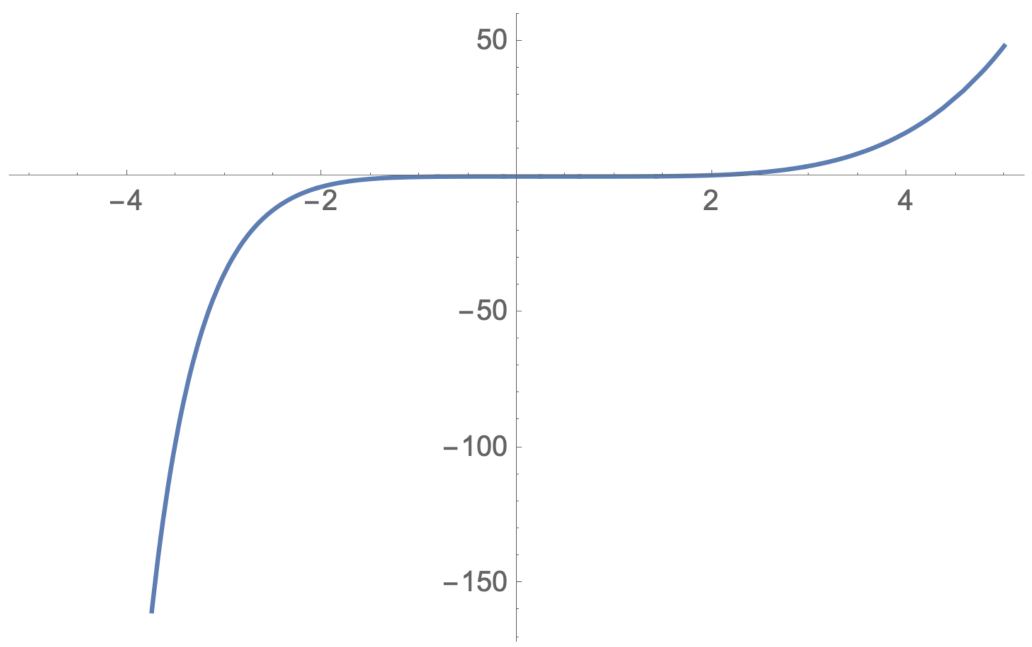
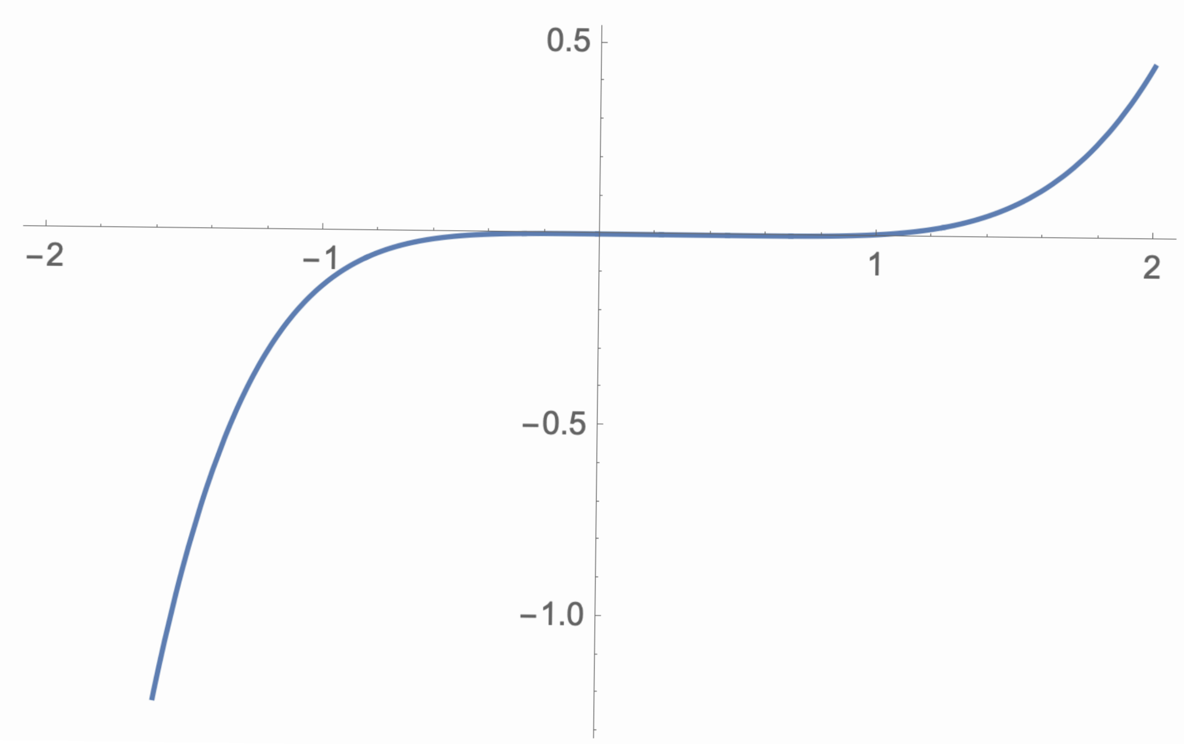
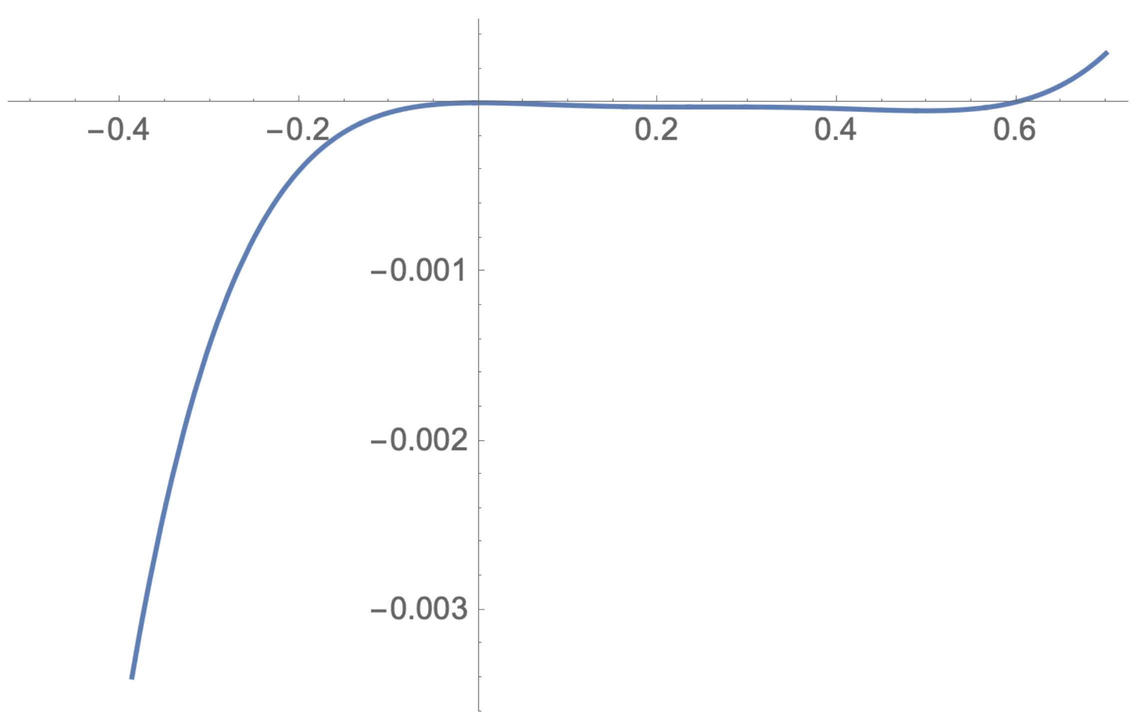
Plot[Evaluate[r[x]], {x, -5, 5}]
```



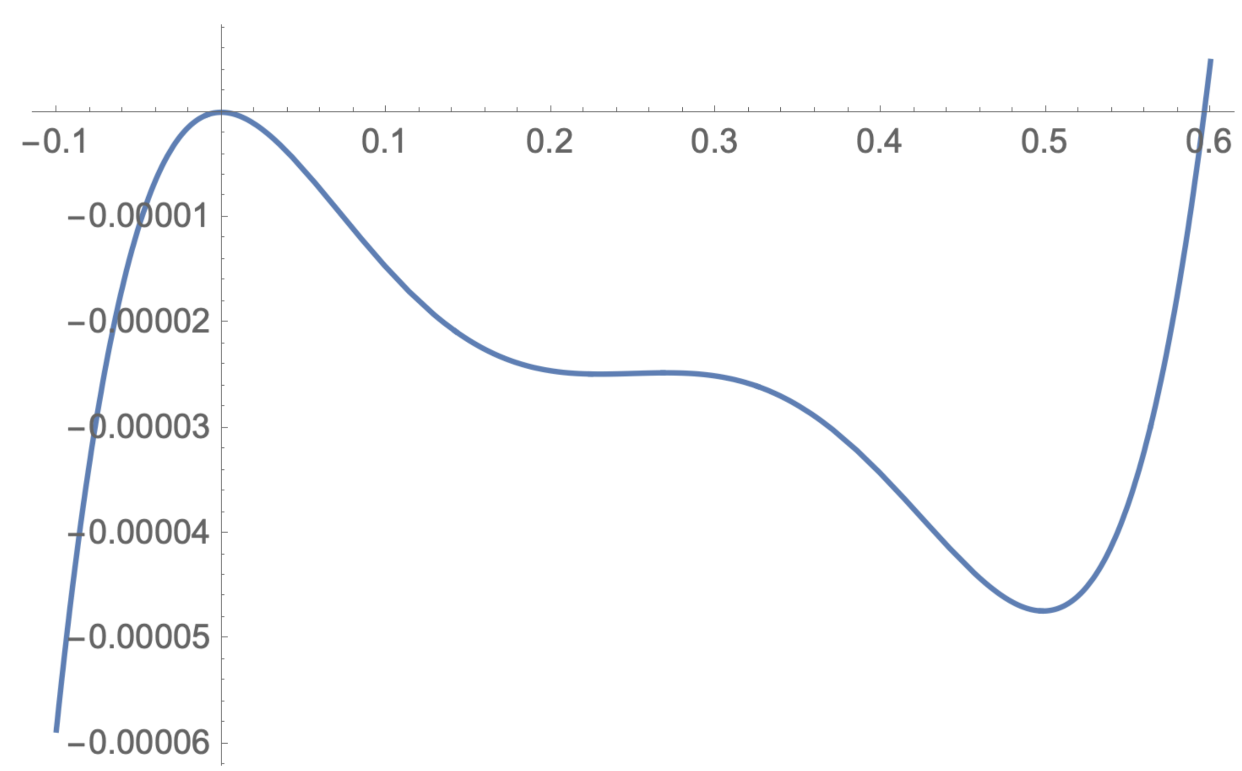

```
Plot[Evaluate[r[x]], {x, -2, 2}]
```




```
Plot[Evaluate[r[x]], {x, -.5, .7}]
```

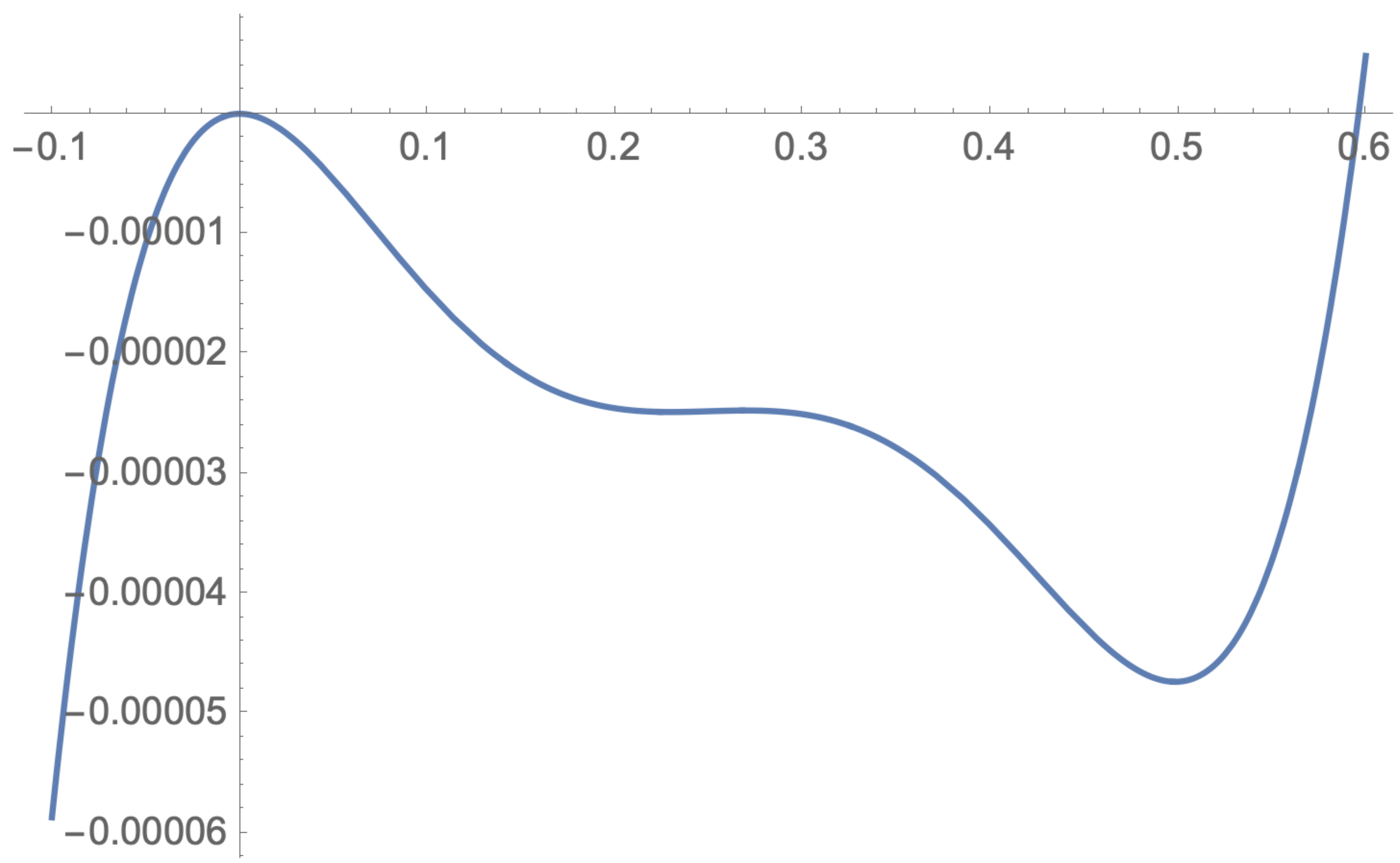



```
Plot[Evaluate[r[x]], {x, -.1, .6}]
```

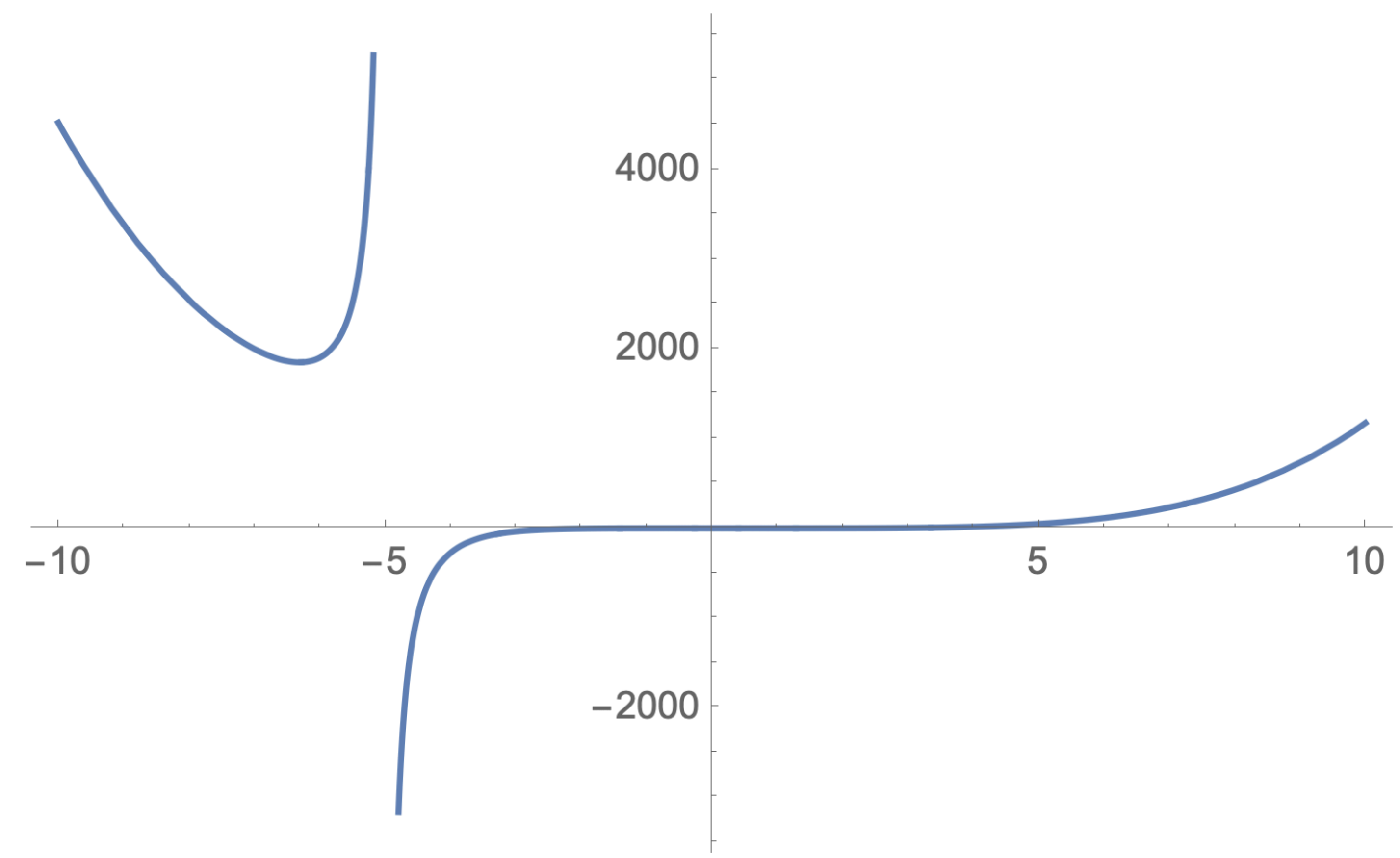


We have computers. Why even bother?

```
Plot[Evaluate[r[x]], {x, -.1, .6}]
```




```
Plot[Evaluate[r[x]], {x, -10, 10}]
```



How do we approach sketching curves using calculus?

① What is the domain?

Avoid zero in denominator, negative #s in $\sqrt{\ }$, negative #s in logs, etc.

② Can we easily find x & y-intercepts?

y-int: evaluate $f(0)$

x-int: solve $f(x) = 0$.

Don't spend much time on this!

③ Are there any horizontal or vertical asymptotes?

HA/long-term behavior: $\lim_{x \to \pm\infty} f(x)$

VA: $\lim_{x \to a^*} f(x)$

④ Where is the function increasing/decreasing?

Critical points: $f'(x) = 0$ or $f'(x)$ DNE

Check sign of f' on either side

⑤ Are there local maxima & minima?

Use increasing/decreasing change

⑥ Where is the function concave up? Concave down? Inflection Points?

Critical points for f':

$f''(x) = 0$ or $f''(x)$ DNE

Test sign of f'' to determine concavity and identify inflection points

⑦ Sketch the curve and label important points

Example: $f(x) = \dfrac{2x^2}{x^2-4}$

① Determine the domain.

Solve denominator $= 0 \implies x^2 - 4 = 0 \implies (x-2)(x+2) = 0$

$\implies x = 2$ or $x = -2$

Domain $= \{x \in \mathbb{R} : x \neq 2,\ x \neq -2\}$

that is, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

② x- and y- intercepts?

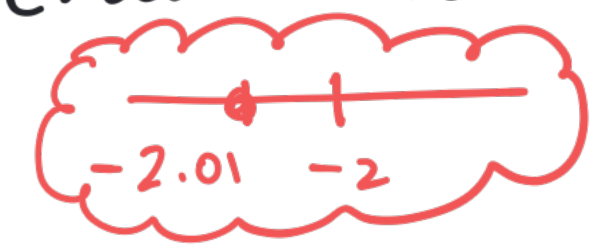
y-int: $f(0) = \dfrac{2(0)^2}{0^2 - 4} = \dfrac{0}{-4} = 0$

x-int: Solve $f(x) = 0$: $\dfrac{2x^2}{x^2-4} = 0 \implies x = 0$

Example: $f(x) = \dfrac{2x^2}{x^2-4} = \dfrac{2x^2}{(x-2)(x+2)}$

③ Asymptotes

Vertical asymptotes: Check behavior near $x=2$, $x=-2$



$$\lim_{x\to -2^-} f(x) : \lim_{x\to -2^-} \frac{\overbrace{2x^2}^{\to 8}}{\underbrace{(x-2)}_{\to -4}\underbrace{(x+2)}_{\to 0^-}} = \infty$$

$$\lim_{x\to 2^+} f(x) : \lim_{x\to 2^+} \frac{\overbrace{2x^2}^{\to 8}}{\underbrace{(x-2)}_{\to 0^+}\underbrace{(x+2)}_{\to 4}} = \infty$$

$x=2$ and $x=-2$ are vertical asymptotes

We can compute $\lim_{x\to -2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ as well, or wait and use other information, like concavity, to determine which direction we approach the asymptote as well!

④ As $x \to \pm\infty$, what does the function do?

$$\lim_{x\to\infty} \frac{2x^2}{x^2-4} = \lim_{x\to\infty} \frac{2}{1-4/x^2} = 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} \frac{2(-x)^2}{(-x)^2-4} = 2$$

$y=2$ is a HA in both directions!

Example: $f(x) = \dfrac{2x^2}{x^2-4}$

Increasing/Decreasing: $f'(x) = \dfrac{(x^2-4)(4x) - (2x^2)(2x)}{(x^2-4)^2}$

$$= \frac{2x\left((x^2-4)(2) - 2x^2\right)}{(x^2-4)^2} = \frac{2x\left(2x^2 - 8 - 2x^2\right)}{(x^2-4)^2} = \frac{2x(-8)}{(x^2-4)^2} = \frac{-16x}{(x^2-4)^2}$$

always +

Critical points:

a) $f'(x) = 0 \Rightarrow -16x = 0 \Rightarrow x = 0$

b) $f'(x)$ DNE $\Rightarrow x = 2,\ x = -2$

x		-2		0		2	
sample	-3		-1		1		3
f'	+	DNE	+	0	−	DNE	−
f	↗		↗	MAX	↘		↘

$f'(-3) = \dfrac{-16(-3)}{((-3)^2-4)^2} = \dfrac{+}{+}$

$f'(-1) = \dfrac{(-16)(-1)}{+} = +$

$f(1) = \dfrac{(-16)(1)}{+} = -$

$f(3) = \dfrac{(-16)(3)}{+} = -$

Example: $f(x) = \dfrac{2x^2}{x^2-4}$ $\qquad f'(x) = \dfrac{-16x}{(x^2-4)^2}$

Concavity and inflection points:

$$f''(x) = \frac{(x^2-4)^2(-16) - (-16x)(2(x^2-4))(2x)}{((x^2-4)^2)^2}$$

$$= \frac{-16\cancel{(x^2-4)}\left[(x^2-4) - x(2x)(2)\right]}{(x^2-4)^{\cancel{4}}} = \frac{-16(x^2-4-4x^2)}{(x^2-4)^3} = \frac{-16(-4-3x^2)}{(x^2-4)^3}$$

$$= \frac{16(3x^2+4)}{(x^2-4)^3}$$

always positive!

No x where $f''(x)=0$, and f'' DNE at $x=2$, $x=-2$

x		-2		2	
Sample	-3		0		3
f''	$+$	DNE	$-$	DNE	$+$
f	$\cup$		$\cap$		$\cup$

$f''(-3) = \dfrac{+}{((-3)^2-4)^3} = +$

$f''(0) = \dfrac{+}{(0-4)^3} = -$

$f''(3) = \dfrac{+}{(9-4)^3} = +$

Example: $f(x) = \dfrac{2x^2}{x^2-4}$

Collect information and sketch function.

- function passes through $(0,0)$
- VA at $x=-2$, $x=2$
- HA at $y=2$

x		-2		0		2	
f'	$+$	DNE	$+$		$-$	DNE	$-$
f''	$+$	DNE	$-$		$-$	DNE	$+$
f				MAX